Physical models for micro and nanosystems

Chapter 4: Introduction to the finite element method

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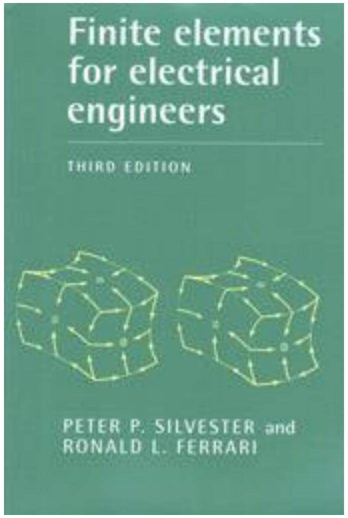
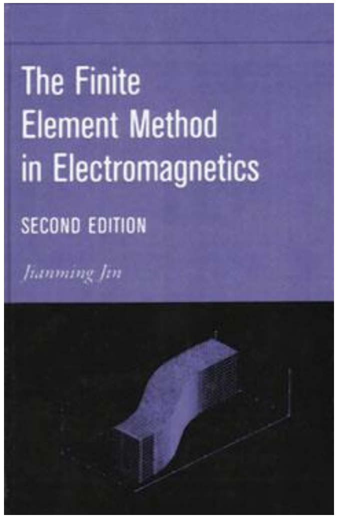
References



Finite Elements for electrical engineers
by Peter P. Silvester and Ronald Ferrari



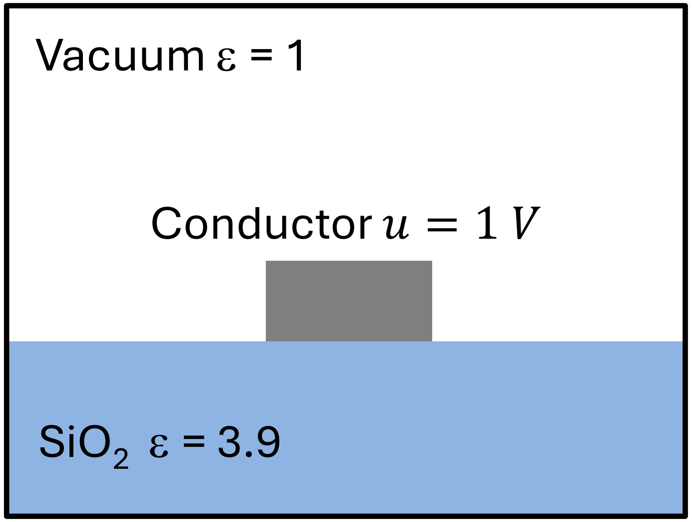
The Finite Element Method in Electromagnetics
by Jianming Jin

Outline

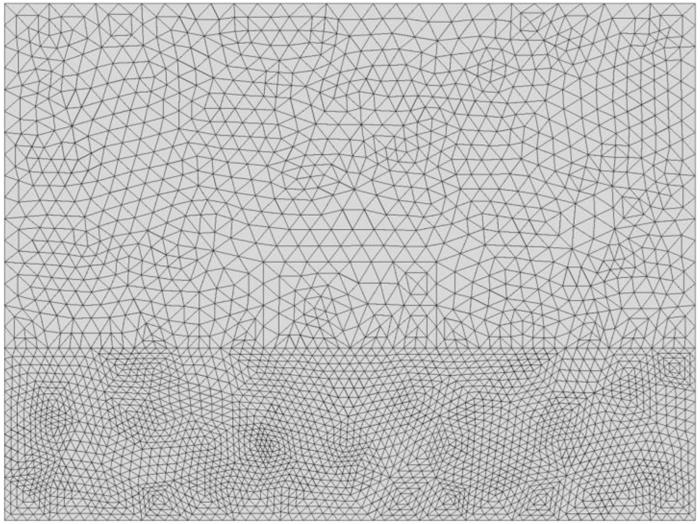
- Illustration of how finite element method works on 1D problems
- Finite elements in 2D: boundary conditions, dealing with interfaces between materials

Example: shielded conductor on a dielectric

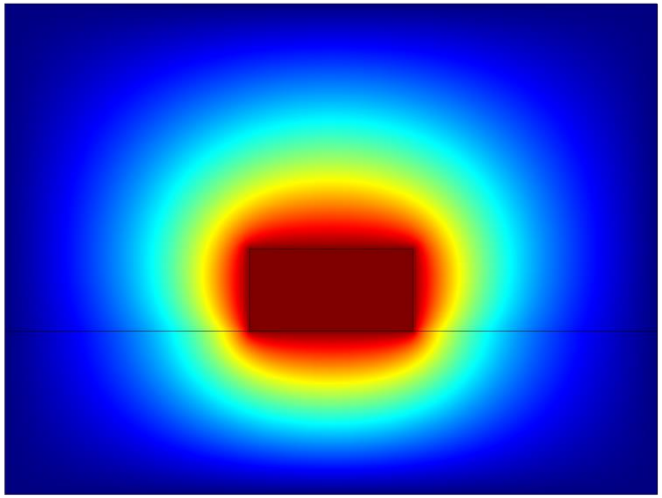
Grounded box
$u = 0\ V$



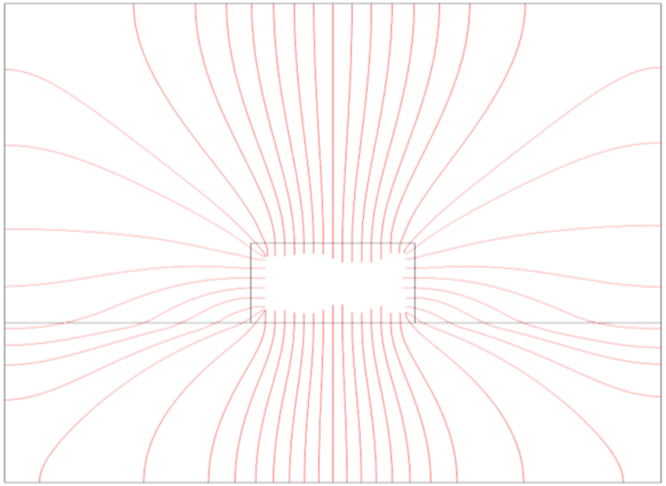
Model



Mesh



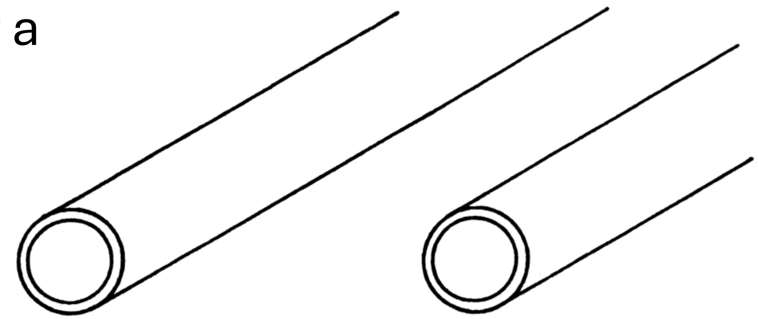
Potential distribution



Electric field lines

Example: DC transmission line (telegraph equations)

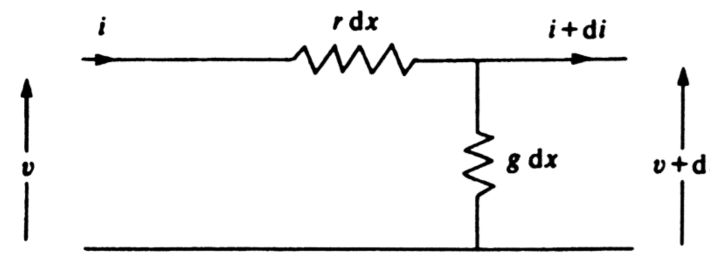
DC current transmission line consisting of a pair of buried pipes



To solve the problem analytically, consider a short length dx

Its logitudinal resistance causes a longitudinal voltage drop dv

Current di flows betweeen pipes, represented by the shunt conductance $g \cdot dx$.



The differences dv and di in voltages and currents at the two ends x and $dx + dx$ are:

$$dv = ir \cdot dx \qquad di = (v + dv)g \cdot dx$$

After rewriting and discarding second-order terms (i.e. $dvg \cdot dx$) these equations become:

$$\frac{dv}{dx} = ri \qquad \frac{di}{dx} = gv$$

Introductory example: DC transmission line

- Differentiating with respect to x:

$$\frac{d^2v}{dx^2} = r\frac{di}{dx} \qquad \frac{d^2i}{dx^2} = g\frac{dv}{dx}$$

Substituting previous two equations into these yields the pair of differential equations:

$$\frac{d^2v}{dx^2} = rgv \qquad \frac{d^2i}{dx^2} = rgi$$

- Two boundary conditions apply here:

- voltage has a known value v_0 at the sending end $x = L$: $v(x = L) = v_0$

- current vanishes at the receiving end: $i(x = 0) = 0$

$$\frac{dv}{dx} = ri$$

- The analytical solution is then:

$$v = v_0 \frac{e^{\sqrt{rg}x} + e^{-\sqrt{rg}x}}{e^{\sqrt{rg}L} + e^{-\sqrt{rg}L}}$$

DC transmission line – the finite element approach

- This approach does not solve the differential equations directly

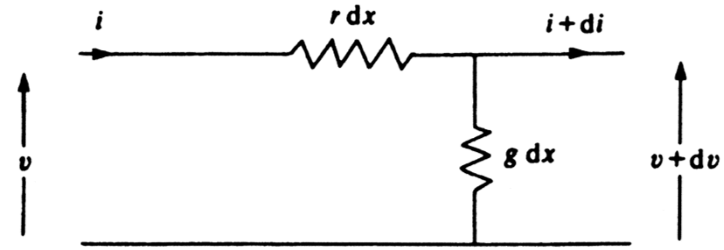
- Instead, a mathematical principle is used which is equivalent to saying that the voltage distribution along the line is such that the power loss is minimized

- This is similar to an approach in analytical (or classical) mechanics where complicated problems can be solved on approaches based on the principle of least action (Euler - Lagrange equations)

Steps in the finite element approach – Step 1/6

- **Express the power** W lost in the line in terms of the voltage distribution $v(x)$:

$$W = W[v(x)]$$



- The power entering any section at its left is:

$$W_{in} = vi$$

- while the power leaving on the right is:

$$W_{out} = (v + dv)(i + di)$$

The difference is the power lost in the section dx. Neglecting second order terms, this difference is given by:

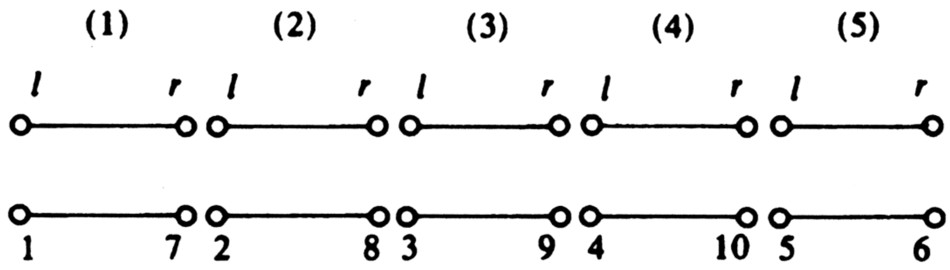
$$dW = W_{in} - W_{out} = -vdi - idv$$

It can be shown (we will not do it here) that the power W lost in the line in terms of the voltage distribution $v(x)$ is:

$$W = -\int_0^L \left[gv^2 + \frac{1}{r}\left(\frac{dv}{dx}\right)^2 \right] dx$$

Step 2/6: subdivision

- **Subdivide** the domain of interest (the entire transmission line) **into K finite sections** (elements)



- The entire line, spanning $0 \leq x \leq L$ is subdivided into K segments (finite elements)

- We number the K elements in order, begining at the left end, label an individual segment with index $k = 1 \dots K$
- Left and right ends will be denoted by suffixes l and r

In this notation, voltages and currents might be labelled as $v_{(k)l}$ or $i_{(k)r}$ and the corresponding positions of the two element ends as $x_{(k)l}$ and $x_{(k)r}$.

Step 3/6: approximate voltage

- We approximate that the voltage varies linearly with distance x within any of the elements. We can then write the following equation:

$$v = \frac{x_{(k)r} - x}{x_{(k)r} - x_{(k)l}} v_{(k)l} + \frac{x - x_{(k)l}}{x_{(k)r} - x_{(k)l}} v_{(k)r}$$

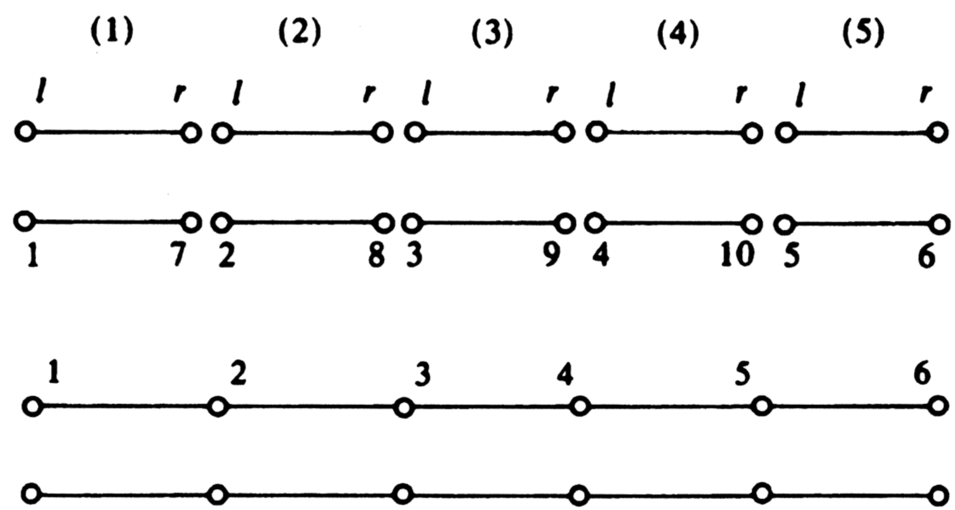
This is a linear combination of voltages on the left and right ends $(v_{(k)l}, v_{(k)r})$

It is convenient to rewrite this as:

$$v = \alpha_l(x) v_l + \alpha_r(x) v_r$$

where

$$\alpha_l(x) = \frac{x_r - x}{x_r - x_l} \qquad \alpha_l(x) = \frac{x - x_l}{x_r - x_l}$$



- **Important condition:** voltages must be continuous across the nodes

Step 4/6: express power lost in each element

- In order to express the power lost in each element, we would have to substitute voltage in this expression:

$$W = -\int_0^L \left[g v^2 + \frac{1}{r}\left(\frac{dv}{dx}\right)^2 \right] dx$$

with:

$$v = \alpha_l(x) v_l + \alpha_r(x) v_r$$

for every segment

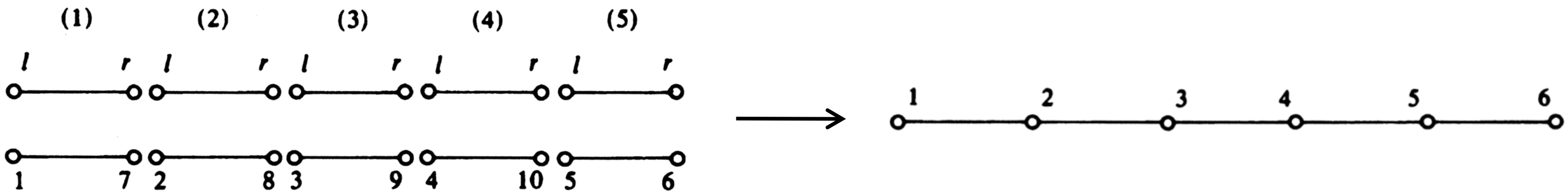
- It can be shown that the power lost in segment k can be written in the following form:

$$W_k = -\begin{bmatrix} v_l & v_r \end{bmatrix} \left(\frac{1}{r_k L_k} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{g_k L_k}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right) \begin{bmatrix} v_l \\ v_r \end{bmatrix}$$

where L_k, r_k and g_k are length, resistance and conductance of segment k.

Step 5/6: Reconnect the elements

- We now have to put together the individual power values for all elements and then minimize power with respect to nodal voltages v_i .



- The total power loss can be written in a matrix form:

$$W = -\mathbf{V}_{con}^T \mathbb{M} \mathbf{V}_{con}$$

where

$$\mathbf{V}_{con} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix}_{con} \qquad \mathbf{V}_{con}^T = [v_1 \quad v_2 \quad v_3 \quad v_4 \quad v_5 \quad v_6]$$

Step 5/6: Reconnect the elements

The total power loss can be written in a matrix form:

$$W = -\mathbf{V}_{con}^{T}\mathbb{M}\mathbf{V}_{con}$$

$$\mathbb{M} = \mathbb{A} + \mathbb{B}$$

$$\mathbb{A} = \frac{1}{L_e r}\begin{bmatrix} 1 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & -1 & 2 & -1 & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 1 \end{bmatrix} \qquad \mathbb{B} = \frac{L_e g}{6}\begin{bmatrix} 2 & 1 & & & & \\ 1 & 4 & 1 & & & \\ & 1 & 4 & 1 & & \\ & & 1 & 4 & 1 & \\ & & & 1 & 4 & 1 \\ & & & & 1 & 2 \end{bmatrix}$$

Step 6/6: Minimize power loss

- We now have to minimize this loss in order to determine the actual values of nodal voltages

- Every nodal voltage is free to vary except for the voltage at the sending end of the line (that one is determined by the source). We can therefore minimize power with respect to all nodal voltages except the first one, so with N nodes we have to differentiate with respect to $N-1$ variables. Hence, power is minimized by setting:

$$\frac{\partial W}{\partial v_k} = 0, \quad k = 1, 2, \dots, N-1$$

Step 6/6: Minimize power loss

On substituting $W = -\mathbf{V}_{con}^T \mathbb{M} \mathbf{V}_{con}$ into $\frac{\partial W}{\partial v_k} = 0$ we get a matrix equation of N columns, one for each node:

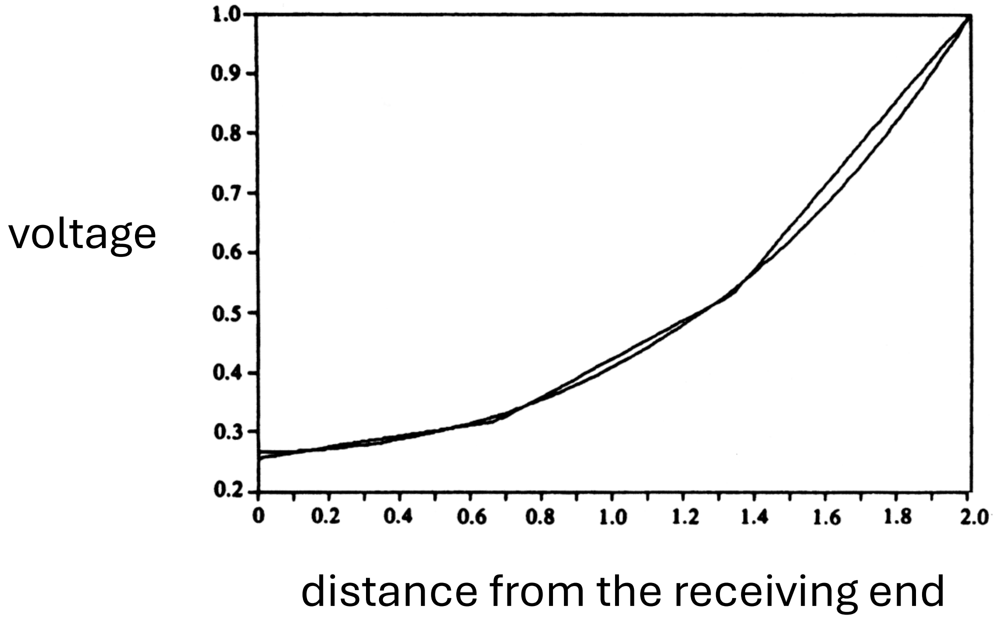
$$\begin{bmatrix} M_{11} & M_{12} & \cdots & & \\ M_{21} & M_{22} & & & \\ \vdots & & \ddots & & \vdots \\ & & & M_{45} & M_{46} \\ & & \cdots & M_{55} & M_{56} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_4 \\ v_6 \end{bmatrix} = 0$$

v_6 is the source voltage and it is known, so we can transpose this voltage to the right hand side:

$$\begin{bmatrix} M_{11} & M_{12} & \cdots & \\ M_{21} & M_{22} & & \\ \vdots & & \ddots & \vdots \\ & & \cdots & M_{55} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_5 \end{bmatrix} = \begin{bmatrix} -M_{16} v_6 \\ -M_{26} v_6 \\ \vdots \\ -M_{56} v_6 \end{bmatrix}$$

Solve as a system of linear equations and we’re done!

Finite element solution



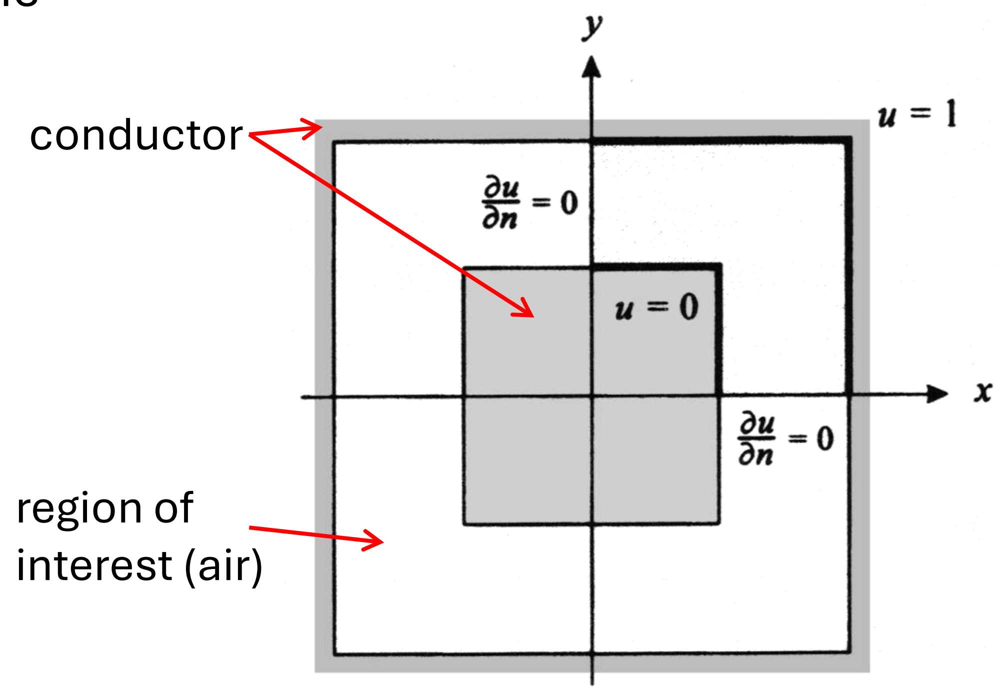
2D problems

Example of a 2D problem

- What is the voltage distribution in the space between conductors in this rectangular coaxial transmission line

- The voltage potential in the space between the conductors is governed by the Laplace equation:

$$\nabla^2 u = 0$$



- As the transmission line has two planes of symmetry, only one quarter of the actual region needs to be analyzed

Boundary conditions in electrostatics

- In electrostatic problems, we are looking for voltage distribution in space
- We can encounter following types of boundary conditions:

1. Known voltage or grounded

   $u = u_0$ Dirichlet boundary conditions

2. Insulation

   $$\sigma \frac{\partial u}{\partial n} = 0 \quad \text{or simply} \quad \frac{\partial u}{\partial n} = 0$$

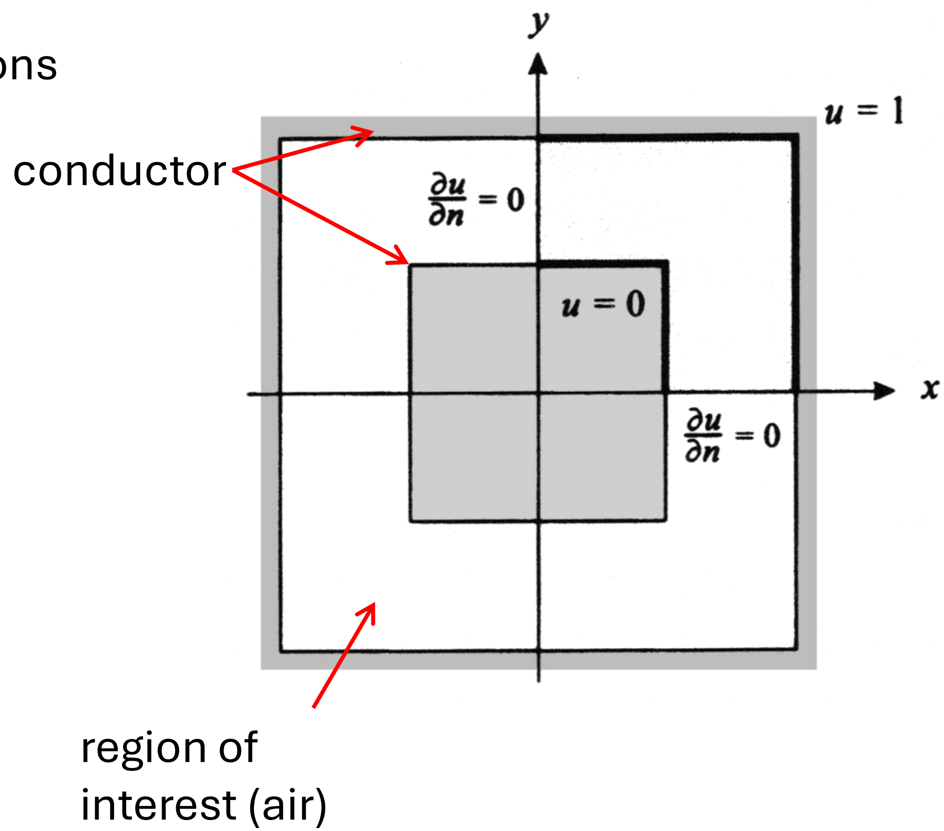
   Von Neumann boundary conditions

3. Symmetry

   $$\frac{\partial u}{\partial n} = 0$$

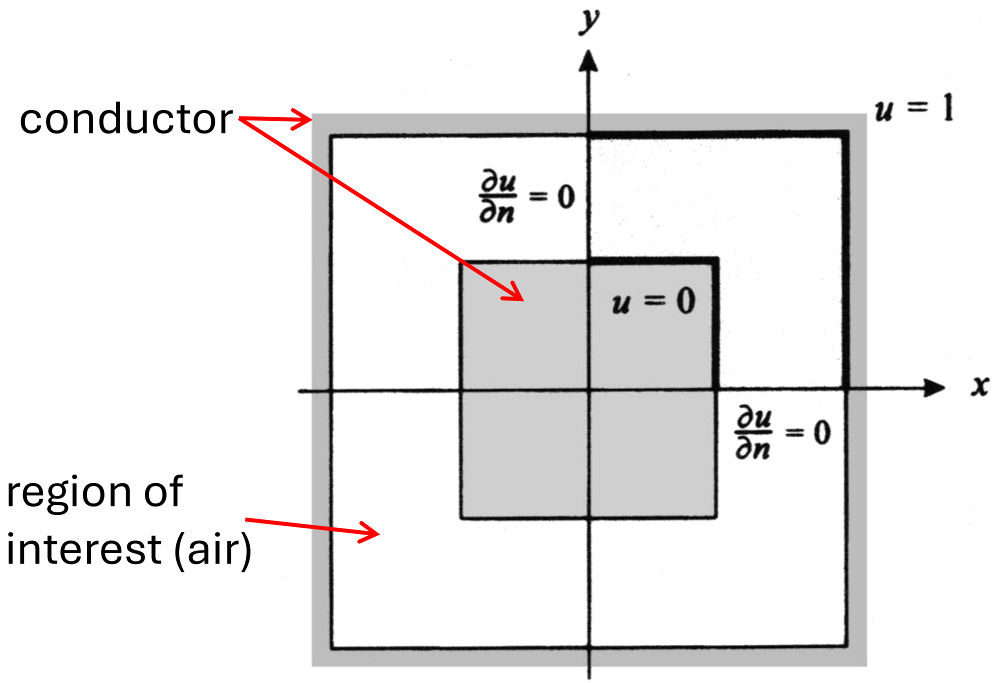
4. Current continuity

   $$\sigma_1 \frac{\partial u_1}{\partial n} = \sigma_2 \frac{\partial u_2}{\partial n}$$

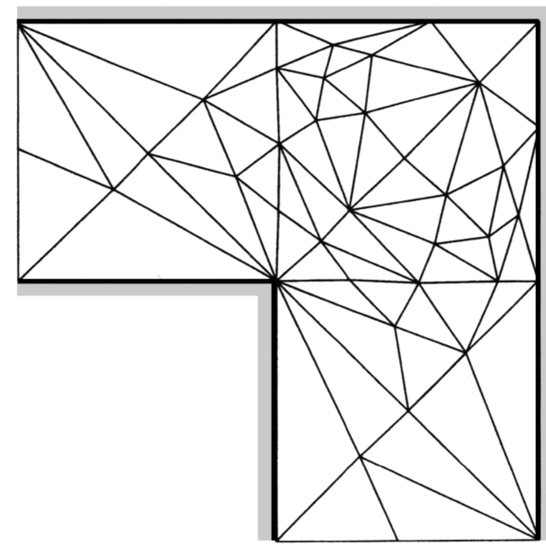


First-order elements

- The problem region is subdivided into triangular elements



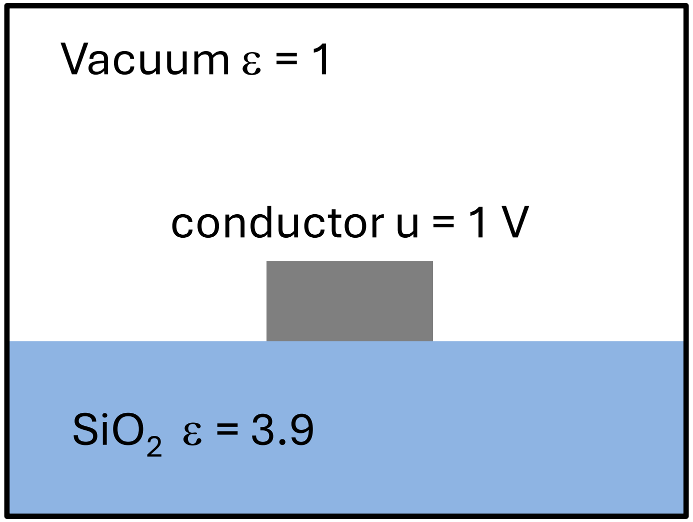
Finite element mesh

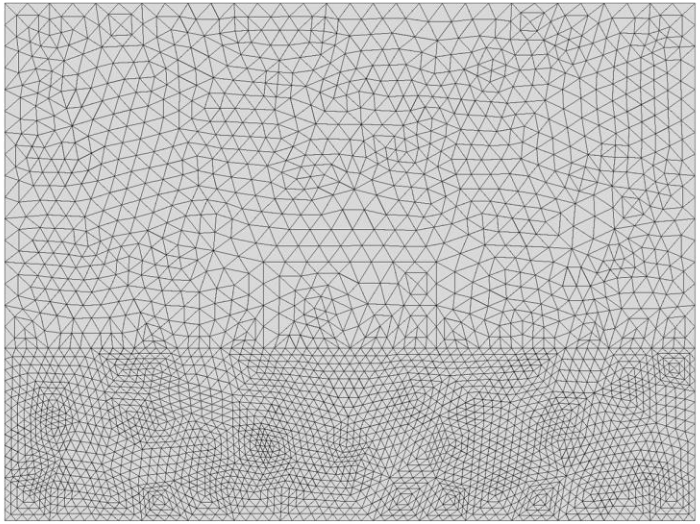


- The finite element method consists of:
  - Approximating the potential u within each element in a standardized fashion
  - Constraining the potential so it becomes continuous across boundaries
- Once the nodal voltages are found, the solution is precisely defined everywhere
- Dirichlet boundary conditions are exactly satisfied, while the von Neumann conditions are satisfied on average

Material inhomogeneities

- Problems involving multiple physical media
- The governing principle is still that of the minimal stored energy
- The finite element method works as before, provided that the element subdivision is such that **element edges follow medium boundaries**
- The **S** matrix (Dirichlet matrix, integrals of the position function) must be multiplied by the local permitivity ε before the element is joined to others





Mesh